where $\boldsymbol{\omega}$ is the vorticity tensor defined by Eq. (2.61). The physical interpretation of the right-hand side of Eq. (2.111) may be given as follows. The first two terms represent the material derivative of a_{ij}, similar to the first two terms on the right-hand side of Eq. (2.104). However, the third and fourth terms containing only the vorticity tensor $\boldsymbol{\omega}$ represent the rotational motion of a material element referred to in a fixed coordinate system. For this reason, the Jaumann derivative is sometimes referred to as the "corotational derivative" (Bird et al. 1987). In Chapter 3 we show that the contravariant and covariant components, respectively, of the Jaumann derivative of the stress tensor give rise to identical expressions for the material functions in steady-state simple shear flow, predicting the same trend as that observed experimentally.

2.6 The Description of Stress and Material Functions

Let us consider now the stress tensor, which causes or arises from deformation. In order to give the reason why a second-order tensor is required to describe the stress, a development of Cauchy's law of motion is needed. The physical significance of the stress tensor may be illustrated best by considering the three forces acting on three faces (one force on each face) of a small cube element of fluid, as schematically shown in Figure 2.4. For instance, a force (which is the vector) acting on the face ABCD with an arbitrary direction may be resolved in three component directions: the force acting in the x_1 direction is $T_{11}dx_2dx_3$, the force acting in the x_2 direction is $T_{12}dx_2dx_3$, and the force acting in the x_3 direction is $T_{13}dx_2dx_3$. Similarly, the forces acting on face BCFE are $T_{21}dx_1dx_3$ in the x_1 direction, $T_{22}dx_1dx_3$ in the x_2 direction, $T_{23}dx_1dx_3$

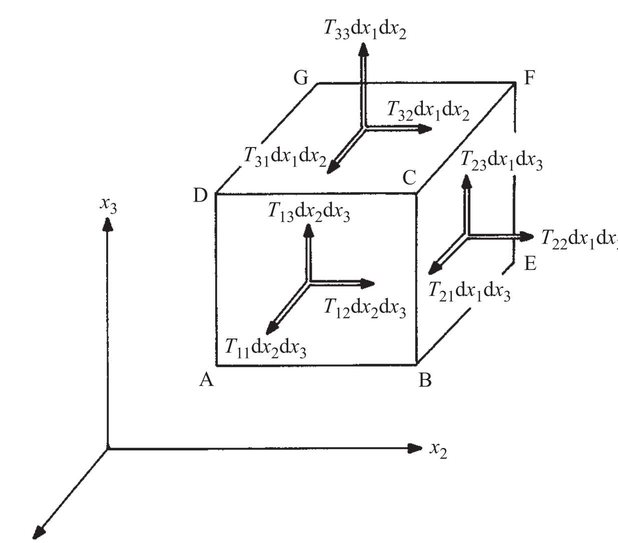

Figure 2.4 Stress components on a cube.

in the x_3 direction. Likewise, the forces acting on face DCFG are $T_{31}\mathrm{d}x_1\mathrm{d}x_2$ in the x_1 direction, $T_{32}\mathrm{d}x_1\mathrm{d}x_2$ in the x_2 direction, and $T_{33}\mathrm{d}x_1\mathrm{d}x_2$ in the x_3 direction.

In dealing with the state of stresses of incompressible fluids under deformation or in flow, the total stress tensor $\mathbf{T}$ is divided into two parts:

$$\left\| T_{ij} \right\| = \left\| \begin{matrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{matrix} \right\| = \left\| \begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} \right\| + \left\| \begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix} \right\| \tag{2.113}$$

where the component T_{ij} of the stress tensor $\mathbf{T}$ is the force acting in the x_i direction on unit area of a surface normal to the x_i direction. The components T_{11}, T_{22}, and T_{33} are called normal stresses since they act normally to surfaces, the mixed components T_{12}, T_{13}, and so on, are called shear stresses. In direct notation, Eq. (2.113), using Cartesian coordinates, can be expressed by

$$\mathbf{T} = -p\boldsymbol{\delta} + \boldsymbol{\sigma} \tag{2.114}$$

where the $\boldsymbol{\delta}$ is the unit tensor, $\boldsymbol{\sigma}$ is the deviatoric stress tensor (or the extra stress tensor) that vanishes in the absence of deformation or flow, and p is the isotropic pressure. Note in Eq. (2.113) or Eq. (2.114) that p has a negative sign since it acts in the direction opposite to a normal stress (T_{11}, T_{22}, T_{33}), which by convention is chosen as pointing out of the cube (see Figure 2.4). It should be mentioned that in an incompressible liquid, the state of stress is determined by the strain or strain history only to within an additive isotropic constant, and thus p appearing in Eq. (2.113) or in Eq. (2.114) is the pressure that can be determined within the accuracy of an isotropic term. As is shown in some later chapters (e.g., Chapter 5), only pressure gradient plays a role in describing fluid motion. Thus the isotropic term, $p\boldsymbol{\delta}$ in Eq. (2.114) has no effect on fluid motion, i.e., the addition of an isotropic term of arbitrary magnitude has no consequence to the total stress tensor $\mathbf{T}$ when a fluid is in motion.

Special types of states of stress are of particular importance. In a liquid that has been at rest (i.e., there is no deformation of a fluid) for a sufficiently long time, there is no tangential component of stress on any plane of a cube and the normal component of stress is the same for all three planes, each perpendicular to the others. This is the situation where only hydrostatic pressure, $-p$, exists. In such a situation, Eq. (2.113) reduces to

$$\left\| T_{ij} \right\| = \left\| \begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} \right\| \tag{2.115}$$

From Eq. (2.115) we can now define pressure as

$$-p = \tfrac{1}{3}(T_{11} + T_{22} + T_{33}) \tag{2.116}$$

Note that Eq. (2.116) can also be obtained from Eq. (2.113) with the assumption, $\sigma_{11} + \sigma_{22} + \sigma_{33} = 0$. Since such an assumption is quite arbitrary, the definition of pressure p given by Eq. (2.116) can be regarded as a somewhat arbitrary one. In fact, in general p is the thermodynamic pressure, which is related to the density ρ and the temperature through a "thermodynamic equations of state," $p = p(\rho, T)$; that is, this is taken to be the same function as that used in thermal equilibrium (Bird et al. 1987).

If we now consider the state of stress in an isotropic material, by definition the material has no preferred directions. In simple shear flow, we have

$$T_{13} = T_{31} = 0; \quad T_{23} = T_{32} = 0; \quad T_{12} = T_{21} \neq 0 \tag{2.117}$$

in which the subscript 1 denotes the direction of flow, the subscript 2 denotes the direction perpendicular to flow, and the subscript 3 denotes the remaining (neutral) direction. It follows therefore from Eq. (2.113) that the most general possible state of stress for an isotropic material in simple shear flow may be represented by

$$\begin{Vmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{Vmatrix} = \begin{Vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{Vmatrix} + \begin{Vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{Vmatrix} \tag{2.118}$$

Note that one cannot measure p and the components of the extra stress tensor $\boldsymbol{\sigma}$ separately during flow of a liquid. Therefore, the absolute value of any one normal component of stress is of no rheological significance. The values of the differences of normal stress components are, however, not altered by the addition of any isotropic pressure (see Eq. (2.118)), and they presumably depend on the rheological properties of the material. It follows, therefore, that there are only three independent stress quantities of rheological significance, namely, one shear component and two differences of normal components:

$$\sigma_{12}; \quad T_{11} - T_{22} = \sigma_{11} - \sigma_{22}; \quad T_{22} - T_{33} = \sigma_{22} - \sigma_{33} \tag{2.119}$$

Note that the normal stress difference $\sigma_{11} - \sigma_{33}$ becomes redundant since we have assumed $\sigma_{11} + \sigma_{22} + \sigma_{33} = 0$ in defining p by Eq. (2.116). In the rheology community, $N_1 = \sigma_{11} - \sigma_{22}$ is referred to as the first normal stress difference and $N_2 = \sigma_{11} - \sigma_{33}$ as the second normal stress difference. It now remains to be discussed how the stress quantities may be related to strain or rate of strain to describe the rheological properties of materials, in particular polymeric materials.

For steady-state shear flow, the components of the stress tensor $\mathbf{T}$ may be expressed in terms of three independent functions:

$$\sigma_{12} = \eta(\dot{\gamma})\dot{\gamma} \quad N_1 = \psi_1(\dot{\gamma})\dot{\gamma}^2 \quad N_2 = \psi_2(\dot{\gamma})\dot{\gamma}^2 \tag{2.120}$$

where $\eta(\dot{\gamma})$ is referred to as the shear-rate dependent viscosity, $\psi_1(\dot{\gamma})$ as the first normal stress difference coefficient, and $\psi_2(\dot{\gamma})$ as the second normal stress difference coefficient. Often, $\eta(\dot{\gamma})$, $\psi_1(\dot{\gamma})$, and $\psi_2(\dot{\gamma})$ are referred to as the "material functions" in steady-state shear flow. Note that N_1 and N_2, or $\psi_1(\dot{\gamma})$ and $\psi_2(\dot{\gamma})$, describe the fluid elasticity, which is elaborated on in Chapter 3.

In the past, numerous investigators have reported measurements of the rheological properties of polymeric liquids. Until now, very few polymeric fluids, if any, which exhibit a constant value of shear viscosity (i.e., $\eta(\dot{\gamma}) = \eta_0$) exhibit measurable values

Symmetry of the Stress Tensor

Symmetry of the Stress Tensor

One further piece of information emerges from applying Newton's law to an infinitely small fluid particle. This is that the stress tensor is in most cases symmetric, that is, $\tau_{ij} = \tau_{ji}$ for $i \neq j$.

The proof follows from considering the angular acceleration of a little fluid particle at x, y, z. For convenience, we let it be shaped like a little cube with sides $\delta x, \delta y$, and δz. Since we shall be taking the limit where $dx, dy, dz \to 0$, where the fluid particle is reduced to a point, we can safely assume that the values of the density, velocity, stress tensor components, etc. are uniform throughout the cube. If the cube has an angular velocity $\dot{\theta} z$ in the z-direction, we know from Newton's law, written in angular momentum form, that at any given instant

$$I_z \frac{\delta\dot{\theta}_z}{dt} = T_z \ , \qquad (12)$$

where

$$\begin{aligned} I_z &= \int_{x-\frac{\delta x}{2}}^{x+\frac{\delta x}{2}} \int_{y-\frac{\delta y}{2}}^{y+\frac{\delta y}{2}} \int_{z-\frac{\delta z}{2}}^{z+\frac{\delta z}{2}} \rho(x^2+y^2)dxdydz \\ &= \frac{\rho}{12}[(\delta x)^2 + (\delta y)^2]dxdydz \qquad (13) \end{aligned}$$

is the appropriate moment of inertia of the cube and T_z is the net torque acting on the cube, taken along an axis running through the center of the cube parallel to the z-axis.

FIGURE 5: Illustration of the reason for the stress tensor's symmetry.

FIGURE 6: x-direction surface stresses acting on a fluid particle.

Figure 5 shows the stresses acting on the cube. On the face with $\mathbf{n} = \mathbf{i}$, for example, there is by definition a stress τ_{xx} in the positive x-direction and a stress τ_{xy} in the positive y-direction, while on the face with $\mathbf{n} = \mathbf{i}$, the corresponding stresses have the same magnitudes but opposite directions [see (11) or (11)]. The net torque about an axis through the cube's center, parallel to the z-axis, is caused by the shear forces (the pressure forces act through the center) and by any volumetric torque exerted by the external body force filed. A body force field like gravity acts through the cube's center and exerts no torque about that point. Let us assume for the sake of generality, however, that the external body force does exert *a torque* $\mathbf{t}$ *per unit volume* at the particle's location. The net torque in the z-direction around the particle's center is then

$$\begin{aligned} T_z &= 2\frac{\delta x}{2}\tau_{xy}\delta y\delta z - 2\frac{\delta x}{2}\tau_{yx}\delta y\delta z + t_z\delta x\delta y\delta z \\ &= (\tau_{xy} - \tau_{yx} + t_z)\delta x\delta y\delta z \ . \qquad (14) \end{aligned}$$

From (12) - (14) we see that

$$\tau_{xy} - \tau_{yx} + t_z = \frac{\rho}{12}\frac{d\dot{\theta}_z}{dt}\left[(\delta x)^2 + (\delta y)^2\right] \ ; \qquad (15)$$

As we approach a point in the fluid by letting $\delta x, \delta y \to 0$, we obtain

$$\tau_{xy} = \tau_{yx} - t_z \ ,$$

or, more generally, the off-diagonal stress tensor components must satisfy or, more generally, the off-diagonal stress tensor components must satisfy

$$\tau_{ij} = \tau_{ji} + t_k \ , \qquad (16)$$

where i, j, k form a right-hand triad (e.g., in cartesian coordinates they are in the order x, y, z, or y, z, x, or z, x, y).

Volumetric body torque can exist in magnetic fluids, for example (e.g., see R. E. Rosensweig, *Ferrohydrodynamics*, 1985, Chapter 8). In what follows we shall *assume that volumetric body torque is absent*, in which case (16) shows that the off-diagonal or shear terms in the stress tensor are *symmetric*,

$$\tau_{ji} = \tau_{ij} \qquad (i \neq j) \ . \qquad (17)$$

This means that three of the nine components of the stress tensor can be derived from the remaining ones; that is, *the stress tensor has only six independent components*.